



## On Two Conjectures Concerning the Partial Sums of the Harmonic Series

Stephen M. Zemyan

*Proceedings of the American Mathematical Society*, Vol. 95, No. 1. (Sep., 1985), pp. 83-86.

Stable URL:

<http://links.jstor.org/sici?sici=0002-9939%28198509%2995%3A1%3C83%3AOTCCTP%3E2.0.CO%3B2-%23>

*Proceedings of the American Mathematical Society* is currently published by American Mathematical Society.

---

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at <http://www.jstor.org/about/terms.html>. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at <http://www.jstor.org/journals/ams.html>.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

---

The JSTOR Archive is a trusted digital repository providing for long-term preservation and access to leading academic journals and scholarly literature from around the world. The Archive is supported by libraries, scholarly societies, publishers, and foundations. It is an initiative of JSTOR, a not-for-profit organization with a mission to help the scholarly community take advantage of advances in technology. For more information regarding JSTOR, please contact [support@jstor.org](mailto:support@jstor.org).

## ON TWO CONJECTURES CONCERNING THE PARTIAL SUMS OF THE HARMONIC SERIES

STEPHEN M. ZEMYAN

ABSTRACT. Let  $S_n$  denote the  $n$ th partial sum of the harmonic series. For a given positive integer  $k > 1$ , there exists a unique integer  $n_k$  such that  $S_{n_k-1} < k < S_{n_k}$ . It has been conjectured that  $n_k$  is equal to the integer nearest  $e^{k-\gamma}$ , where  $\gamma$  is Euler's constant. We provide an estimate on  $n_k$  which suggests that this conjecture may have to be modified. We also propose a conjecture concerning the amount by which  $S_{n_k-1}$  and  $S_{n_k}$  differ from  $k$ .

### 1. The first conjecture. Let

$$(1) \quad S_n = \sum_{m=1}^n \frac{1}{m}$$

denote the  $n$ th partial sum of the harmonic series. It is a well-known fact (see [7, pp. 380–381]) that  $S_n$  is never an integer for  $n > 1$ . Since the harmonic series diverges, there exists for each integer  $k > 1$  a corresponding integer  $n_k$  such that

$$(2) \quad S_{n_k-1} < k < S_{n_k}.$$

The question we wish to address here is: What is the value of  $n_k$  for each  $k > 1$ ? It has been conjectured in *Advanced Problems* 5346 (1967, p. 209) and 5989 (1974, p. 910) in the *American Mathematical Monthly* that  $n_k$  is equal to  $((e^{k-\gamma}))$ , where  $\gamma = 0.5772156649 \dots$  is the Euler-Mascheroni constant and  $((x))$  denotes the integer nearest  $x$ . L. Comtet [6, p. 209] shows that either  $n_k = [e^{k-\gamma}]$  or  $[e^{k-\gamma}] + 1$ , where  $[x]$  denotes the greatest integer  $\leq x$ . R. P. Boas comments [2, p. 749] in a partial solution to this problem that this conjecture has been verified by computation for  $k \leq 200$  by Boas (unpublished) and for  $k \leq 1000$  by R. Spira (unpublished). Boas also establishes [5, p. 865] the following partial result: If  $m$  and  $\delta$  are the integral and fractional parts of  $e^{k-\gamma}$ , then  $n_k = ((e^{k-\gamma}))$  provided that  $\delta \notin [1/2 - 1/10m, 1/2 + 1/m]$ .

The following result suggests that the above conjecture may possibly be improved.

**THEOREM.** *For each integer  $k > 1$ , the integer  $n_k$  satisfies the inequality*

$$(3) \quad -\frac{2}{e^{3(k-\gamma)}} - \frac{1}{2} < n_k - \left( e^{k-\gamma} - \frac{1}{24e^{k-\gamma}} \right) < +\frac{1}{2} + \frac{2}{e^{3(k-\gamma)}}.$$

---

Received by the editors June 6, 1984 and, in revised form, October 15, 1984. Presented to the Society at the 90th Annual Meeting in Louisville, Kentucky on January 25, 1984.

1980 *Mathematics Subject Classification.* Primary 40A05.

REMARK. This theorem suggests that  $n_k$  may be equal to the integer nearest  $e^{k-\gamma} - 1/(24e^{k-\gamma})$ . On one hand, the original conjecture could perhaps be modified to reflect the new information contained in the above theorem. On the other hand, this theorem also suggests that we are not yet in a position to postulate a "correct" conjecture as to the value of  $n_k$ .

Before proving this theorem, we digress in order to pose the following question: Is it true that

$$(4) \quad ((e^{k-\gamma})) = \left( \left( e^{k-\gamma} - \frac{1}{24e^{k-\gamma}} \right) \right)$$

for every integer  $k > 1$ ? More generally, for which positive integers  $N$  and real numbers  $\alpha$  is it true that

$$(5) \quad ((e^{k-\alpha})) = \left( \left( e^{k-\alpha} - \frac{1}{Ne^{k-\alpha}} \right) \right)$$

for all integers  $k > 1$ ? If (5) holds for some fixed  $N$  and all  $k > 1$ , must  $\alpha$  be transcendental? Let  $A$  denote the set of all  $\alpha$  for which (5) holds for some fixed integer  $N$  and all integers  $k > 1$ . Is the Lebesgue measure of  $A$  equal to zero?

PROOF OF THE THEOREM. We first establish some background information. It is well known that

$$(6) \quad \lim_{n \rightarrow \infty} \left( \sum_{m=1}^n \frac{1}{m} - \ln n \right) = \gamma.$$

If we define, for  $n \geq 1$ ,

$$(7) \quad \delta(n) = \sum_{m=1}^n \frac{1}{m} - \ln n - \gamma,$$

then it is easily shown that  $\delta(n) \downarrow 0$  as  $n \rightarrow \infty$ . An application of the Euler-Maclaurin Summation Formula (see [1, p. 444, 4, pp. 261–262 or 3, p. 256]) yields the estimate

$$(8) \quad \delta(n) = 1/(2n) - 1/(12n^2) + R_n$$

where  $0 < R_n < 1/(120n^4)$ .

Now define the rational number  $r_k$  by the relation

$$(9) \quad k = 1 + \frac{1}{2} + \cdots + \frac{1}{n_k - 1} + \frac{r_k}{n_k}.$$

Note especially that  $0 < r_k < 1$ . Using (7) and (9), we obtain the relation

$$(10) \quad \ln n_k = k - \gamma - \delta(n_k) + (1 - r_k)/n_k$$

which may be exponentiated to obtain  $n_k = e^{k-\gamma} e^{\beta(n_k)}$ , where we have set

$$(11) \quad \beta(n_k) = -\delta(n_k) + \frac{1 - r_k}{n_k} = \left( \frac{1}{2} - r_k \right) \frac{1}{n_k} + \frac{1}{12n_k^2} - R_{n_k}.$$

Consequently, we have

$$(12) \quad n_k - e^{k-\gamma} = e^{k-\gamma} (e^{\beta(n_k)} - 1).$$

We first show that

$$(13) \quad -1 < n_k - e^{k-\gamma} < +1.$$

Since  $\delta(n) > 0$  for all  $n \geq 1$ , we have

$$\ln(n_k - 1) < \ln(n_k - 1) + \delta(n_k - 1) = S_{n_k-1} - \gamma < k - \gamma.$$

Exponentiating, we obtain the right-hand side of (13). Since  $\ln(n + 1) - \ln(n) > 1/(n + 1) > \delta(n)$  for all  $n \geq 1$ , we have

$$\ln(n_k + 1) > \ln(n_k) + \delta(n_k) = S_{n_k} - \gamma > k - \gamma.$$

Exponentiating, we obtain the left-hand side of (13).

Now let  $n_k - e^{k-\gamma} = a$ . From (12) we obtain the equation

$$(14) \quad \beta(n_k) = \ln\left(1 + \frac{a}{e^{k-\gamma}}\right).$$

Substituting (11) into (14) and solving for  $r_k$ , we get

$$r_k = \frac{1}{2} + \frac{1}{12n_k} - n_k R_{n_k} - a \ln\left(1 + \frac{a}{e^{k-\gamma}}\right) - e^{k-\gamma} \ln\left(1 + \frac{a}{e^{k-\gamma}}\right).$$

Now, since  $|a| < 1$ , we may write

$$-a \ln\left(1 + \frac{a}{e^{k-\gamma}}\right) = -\frac{a^2}{e^{k-\gamma}} + \frac{a^3}{2e^{2(k-\gamma)}} + \frac{c_1}{e^{3(k-\gamma)}},$$

where  $|c_1| \leq 4/9$ , and

$$-e^{k-\gamma} \ln\left(1 + \frac{a}{e^{k-\gamma}}\right) = -a + \frac{a^2}{2e^{k-\gamma}} - \frac{a^3}{3e^{2(k-\gamma)}} + \frac{c_2}{e^{3(k-\gamma)}}$$

where  $|c_2| \leq 1/3$ . Also,

$$(15) \quad \frac{1}{12n_k} = \frac{1}{12e^{k-\gamma}} - \frac{a}{12e^{2(k-\gamma)}} + \frac{c_3}{e^{3(k-\gamma)}}$$

where  $|c_3| \leq 1/9$ , and

$$0 < n_k R_{n_k} < \frac{1}{120n_k^3} < \frac{1}{9e^{3(k-\gamma)}}.$$

The bounds on  $c_1$ ,  $c_2$  and  $c_3$  are independent of  $k$ . Hence,

$$(16) \quad r_k = \left(\frac{1}{2} + \frac{1}{12e^{k-\gamma}} + \frac{p}{e^{3(k-\gamma)}}\right) - a\left(1 + \frac{1}{12e^{2(k-\gamma)}}\right) - \frac{a^2}{2e^{k-\gamma}} + \frac{a^3}{6e^{2(k-\gamma)}}$$

where  $|p| < 1$ . This last expression is a polynomial in  $a$  from which the theorem is easily deduced using the inequality  $0 < r_k < 1$ .

**2. A second conjecture.** Let us again consider inequality (2) and ask the following question: By how much must the closest partial sums miss a given integer  $k$ ? More precisely, can one determine functions  $\alpha$  and  $\beta$  of  $k$  alone such that

$$(17) \quad S_{n_k} - k \geq \alpha(k)$$

and

$$(18) \quad k - S_{n_k-1} \geq \beta(k)?$$

Recalling equation (9), we get

$$(19) \quad S_{n_k} - k = (1 - r_k)/n_k$$

and

$$(20) \quad k - S_{n_k-1} = r_k/n_k.$$

Thus, any lower estimate of the differences in (17) or (18) will involve estimates of  $r_k$  and  $n_k$ .

Suppose now that the original conjecture about the value of  $n_k$  is true; i.e., that  $|n_k - e^{k-\gamma}| = |a| < 1/2$ . (It was noted earlier in this paper that the conjecture is true for values of  $k \leq 1000$ .) If  $a > -1/2$ , then equation (16) implies that

$$(21) \quad 1 - r_k > \frac{1}{24e^{k-\gamma}} + \frac{1}{16e^{2(k-\gamma)}} + \frac{p}{e^{3(k-\gamma)}},$$

where  $|p| < 1$ . Combining (15), (17), (19) and (21), we hypothesize that we may choose

$$\alpha(k) = 1/24e^{2(k-\gamma)} + 1/48e^{3(k-\gamma)}$$

and that  $1/24$  is the best possible constant here. The author is unable at this time to propose an analogous hypothesis concerning the order of magnitude of  $\beta(k)$ , but it seems reasonable to suggest that  $\beta(k) = O(e^{-2k})$ .

#### REFERENCES

1. T. M. Apostol, *Calculus*, Blaisdell, New York, 1962.
2. R. P. Boas, *Partial solution to advanced problem 5989\**, Amer. Math. Monthly **83** (1976), 749.
3. ———, *Partial sums of infinite series, and how they grow*, Amer. Math. Monthly **84** (1977), 237–258.
4. ———, *Growth of partial sums of divergent series*, Math. Comp. **31** (1977), 257–264.
5. R. P. Boas and J. W. Wrench, Jr., *Partial sums of the harmonic series*, Amer. Math. Monthly **78** (1971), 864–870.
6. L. Comtet, *Problem 5346*, Amer. Math. Monthly **74** (1967), 209.
7. G. Polya and G. Szegő, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, Springer-Verlag, Berlin, 1925, pp. 159, 380–381; problems VII, pp. 250, 251.

DEPARTMENT OF MATHEMATICS, THE PENNSYLVANIA STATE UNIVERSITY, MONT ALTO, PENNSYLVANIA 17237

## LINKED CITATIONS

- Page 1 of 1 -



You have printed the following article:

### **On Two Conjectures Concerning the Partial Sums of the Harmonic Series**

Stephen M. Zemyan

*Proceedings of the American Mathematical Society*, Vol. 95, No. 1. (Sep., 1985), pp. 83-86.

Stable URL:

<http://links.jstor.org/sici?sici=0002-9939%28198509%2995%3A1%3C83%3AOTCCTP%3E2.0.CO%3B2-%23>

---

*This article references the following linked citations. If you are trying to access articles from an off-campus location, you may be required to first logon via your library web site to access JSTOR. Please visit your library's website or contact a librarian to learn about options for remote access to JSTOR.*

## **References**

### <sup>4</sup> **Growth of Partial Sums of Divergent Series**

R. P. Boas, Jr.

*Mathematics of Computation*, Vol. 31, No. 137. (Jan., 1977), pp. 257-264.

Stable URL:

<http://links.jstor.org/sici?sici=0025-5718%28197701%2931%3A137%3C257%3AGOPSOD%3E2.0.CO%3B2-G>

**NOTE:** *The reference numbering from the original has been maintained in this citation list.*